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A Kalman-filter based time-domain analysis for structural damage diagnosis with noisy signals

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Abstract

In this paper, a procedure is presented for the time-domain analysis of noise-contaminated vibration signals for global structural damage diagnosis. It extends from a previously established acceleration response-only time-domain Auto-Regressive- with-eXogenous input (ARX) model, where the "process" is defined such that the acceleration response at a given degree of freedom (dof) is regarded as the "input", while the accelerations at other dofs are the "state" with which the "measurements" are associated. The novel idea in the present procedure is to retrieve the intrinsic input–output set from noisy signals by using the Kalman filter, so that the underlying physical system is best presented to the subsequent diagnosis operation. The theoretical basis of representing the system by pairing the raw measured input and the filtered response through the Kalman filter is discussed. When such raw input and filtered response signals are fed into the reference ARX model, the error feature becomes indicative of the change of the physical system. By analyzing the residual error, the damage status of the structure can be diagnosed. Applications to numerical and experimental examples demonstrate that the approach is effective in tackling the noises, and both the occurrence and relative extent of damage can be assessed with an appropriate damage feature.

1. Introduction

In the process of acquiring vibration signals from physical testing, a certain level of noise contamination is inevitable. Subsequent analysis based on the measured signals will more or less be affected by the noise contents in the signals. This problem could become more significant when performing model estimation and signal prediction in the time domain. In the time-domain model estimation stage, some statistic features can be altered by the noise; consequently the model parameters could be wrongly estimated. In the signal prediction stage, the noise in the measurement certainly affects the performance of the model.

A time-domain analysis approach based on Auto-Regressive with eXogenous input (ARX) was proposed in a previous paper by the authors [1] for global structural damage diagnosis. It was shown that under certain conditions, a process can be established with the acceleration at a particular degree of freedom (dof) as the "input" while the accelerations at other measured dofs as the "measurements" (or responses). The model

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Nomenclature

$\mathbf{y}(k)$	output time-series
$\hat{\mathbf{y}}$, $\hat{\mathbf{y}}(k\|k-1)$	predicted output
$\hat{\mathbf{y}}_e$	predicted output by reference model
$\mathbf{y}_{\mathrm{v}}(k)$	measured output with noise
$\mathbf{y}_{\mathrm{vir}}(k)$	virtual output
$\mathbf{y}_{\mathrm{v,vir}}(k)$	virtual output with virtual noise
$\mathbf{x}(k)$	state-vector time series
$\hat{\mathbf{x}}(k\|k)$	posteriori predicted state-vector
$\hat{\mathbf{x}}(k\|k-1)$	priori predicted state-vector
$\mathbf{u}(k)$	input time-series
$\mathbf{u}_{\mathrm{v}}(k)$	measured input with noise
$\mathbf{e}(k)$	residual error time-series
q	shift operator
$\boldsymbol{\eta}(k)$	process noise
$\boldsymbol{\nu}(k)$	measurement noise
$\boldsymbol{\gamma}(k)$	measurement noise of virtual output
$\boldsymbol{\Lambda}$	diagonal matrix
σ	standard deviation
$\mathbf{A}$	state matrix
$A(q)$	auto-regressive polynomial
$\mathbf{B}$	input matrix
$B(q)$	exogenous input polynomial
$\mathbf{C}$	output influence matrix
$\mathbf{D}$	direct transmission matrix
$\mathbf{E}_0, \mathbf{E}_1, \ldots \mathbf{E}_{\mathrm{i}}$	input coefficients in ARMAX
$\mathbf{G}$	input coefficients for process noise
$\mathbf{H}(k)$	covariance of $\boldsymbol{\gamma}(k)$
$\mathbf{H}_1, \mathbf{H}_2, \ldots \mathbf{H}_{\mathrm{i}}$	moving average coefficients in ARMAX
$\mathbf{K}(k)$	Kalman gain
$\mathbf{K}_1, \mathbf{K}_2, \ldots \mathbf{K}_{\mathrm{i}}$	coefficients in Kalman gain $\bar{\mathbf{K}}$
$\bar{\mathbf{K}}$	Kalman gain
$\mathbf{M}(k)$	posteriori estimate error covariance
$\mathbf{P}(k\|k)$	priori estimate error covariance
$\mathbf{P}(k\|k)-1$	posteriori estimate of priori error covariance
$\bar{\mathbf{P}}$	converge of priori error
$\mathbf{P}_1, \mathbf{P}_2, \ldots \mathbf{P}_{\mathrm{i}}$	auto-regression coefficients
$\mathbf{Q}$, $\mathbf{Q}(k)$	process noise covariance
$\mathbf{Q}_1, \mathbf{Q}_2, \ldots \mathbf{Q}_{\mathrm{i}}$	coefficients in input matrix
$\mathbf{R}$, $\mathbf{R}(k)$	measurement noise covariance

coefficients are related to the dynamic properties of the structural system under consideration. A reference model can be established using measurements from a reference state (preferably the original state) of the structure. Relative changes of the structural conditions with respect to the reference state can be diagnosed by feeding the current measurements to the reference model and analyzing the residual error features. This diagnosis procedure can be depicted by a block diagram in Fig. 1.

The approach was shown to work well under noise-free measurement conditions in detecting and locating the damage. However, the performance of the model would deteriorate under noisy measurement conditions, apparently due to the errors introduced by the noise in both the reference model estimation stage and the signal prediction stage. In order to improve the situation, both aspects of the problem need to be addressed. At this juncture, it is important to note that the current damage diagnosis is based on the relative changes of the system; as such, the essential objective in processing the signals from the noisy measurements is to preserve the intrinsic input–output relationship that represents the underlying physical system at the current state. Therefore, processing the signals as a system rather than individual pieces out of the noisy measurements becomes the most important consideration in the current effort to minimize the effect of noise.

For the general purpose of minimizing the noise effect, some methods have been proposed in the past for application in the analysis of autoregressive models. One method is to increase the model order. This method can increase the accuracy of the model in a noisy condition, as shown in several previous studies [2–5], and it is

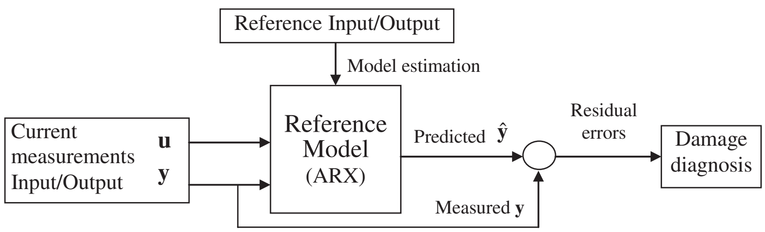


Fig. 1. ARX model-based procedure for damage diagnosis.

commonly used in identifying the dynamic characteristics. However, due to the use of a high order model, extra spurious modes will affect the accuracy of prediction. In this regard, the modal assurance criteria (MAC) or coordinate MAC is usually applied to help divide structural modes and extra spurious modes. Another method is using subspace models for identification [6]. The methods based on subspace models with discrete filter are applicable for linear structures and recordings with wide-band measurement noise, which is usually the case in real life situations. Among the subspace models, the Kalman filter and extended Kalman filter (EFK) are widely adopted in studying time-domain signals [7–12].

In the present paper, the ARX model proposed in the previous study is applied in conjunction with the Kalman filter technique to perform the structural damage diagnosis with noisy measurement signals. The ARX model is first expressed in a state-space form, so that the noise terms can be introduced in to evaluate their effects on the model performance. The expression of the model in a state-space form facilitates the application of the Kalman filter. It is demonstrated on a theoretic basis that using the raw input signal as the Kalman filter input to process the measured response signals, the raw input and the processed response signals gives rise to a desired input–output set that well represent the underlying state of the structure. This input–output signal set can then be fed to the reference model for damage diagnosis using appropriate residual error features.

2. Overview of the Kalman filter

The Kalman filter is an effective tool for stochastic estimation of the state from noisy measurements. Because of its relative simplicity and robust nature, the Kalman filter has been widely used to obtain estimates of the state variables in practice.

The Kalman filter is essentially a set of mathematical equations [13] that aims at minimizing the estimated error covariance in the state estimator. The Kalman Filter proceeds with a given process and measurement equations as

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{G}\boldsymbol{\eta}(k),\\ \mathbf{y}_v(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k),\end{aligned} \tag{1}$$

where $\mathbf{u}(k)$ is the process input of the system, $\mathbf{y}_v(k)$ represents the measurement (output), $\boldsymbol{\eta}(k)$ and $\boldsymbol{\nu}(k)$ are input noise and output noise respectively (they are also called "process noise" and "measurement noise" or "sensor noise"). $\mathbf{G}$ is the input coefficient for the process noise. In most common cases the process noise is introduced into the system together with the input signal, and in such cases $\mathbf{G}$ is equal to $\mathbf{B}$.

The algorithm of the Kalman filter effectively resembles that of a prediction-correction algorithm. The filter estimates the process state and obtains the feedback from the measurement which is noisy. The Kalman filter thus consists of two groups of equations, one performs the prediction (called "time update"), by which the *a priori* estimates for the current step is obtained based on the previous state and error covariance, and the other does the correction (called "measurement update") to obtain an improved *a posteriori* estimate. These equations are recursive nature and can be written [13,14] as:

Time update equations:

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{A}\hat{\mathbf{x}}(k-1|k-1) + \mathbf{B}\mathbf{u}(k-1), \tag{2}$$

$$\mathbf{P}(k|k-1) = \mathbf{A}\mathbf{P}(k-1|k-1)\mathbf{A}^{\mathrm{T}} + \mathbf{G}\mathbf{Q}(k-1)\mathbf{G}^{\mathrm{T}}, \tag{3}$$

$$\mathbf{Q}(k) = E(\boldsymbol{\eta}(k-1)\boldsymbol{\eta}(k-1)^{\mathrm{T}}), \tag{4}$$

$$\mathbf{R}(k) = E(\mathbf{v}(k-1)\mathbf{v}(k-1)^{\mathrm{T}}). \tag{5}$$

Measurement update equations:

$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{M}(k)\big(\mathbf{y}_v(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\big), \tag{6}$$

$$\mathbf{M}(k) = \mathbf{P}(k|k-1)\mathbf{C}^{\mathrm{T}}\big(\mathbf{R}(k) + \mathbf{C}\mathbf{P}(k|k-1)\mathbf{C}^{\mathrm{T}}\big)^{-1}, \tag{7}$$

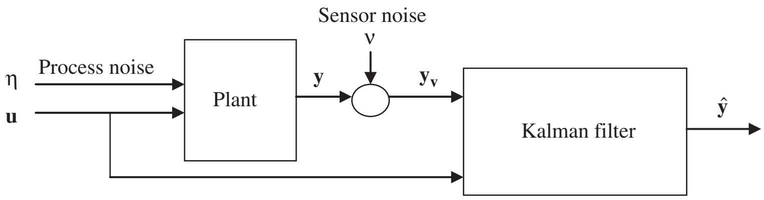


Fig. 2. Typical procedure of stochastic processing of signals with Kalman filter.

$$\mathbf{P}(k|k) = (\mathbf{I} - \mathbf{M}(k)\mathbf{C})\mathbf{P}(k|k-1), \tag{8}$$

with Kalman Gain given by

$$\mathbf{K}(k) = \mathbf{A}\mathbf{M}(k). \tag{9}$$

Combining Eqs. (2) and (6) yields

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{A}\hat{\mathbf{x}}(k-1|k-2) + \mathbf{K}(k-1)\big(\mathbf{y}_v(k-1) - \mathbf{C}\hat{\mathbf{x}}(k-1|k-2)\big) + \mathbf{Bu}(k-1) \tag{10}$$

In these equations, $\hat{\mathbf{x}}$ denotes the estimated process state, $\mathbf{P}$ represents the *a priori* estimate error covariance, $\mathbf{M}$ represents the *a posteriori* estimate error covariance. Given initial conditions $\hat{\mathbf{x}}(1|0)$ and $\mathbf{P}(1|0)$, one can iterate these equations to perform the filtering. The filter can produce an optimal estimate of the true response measurements, $\hat{\mathbf{y}}_e = \hat{\mathbf{y}}(k|k-1)$, by the following equation:

$$\hat{\mathbf{y}}(k|k-1) = \mathbf{C}\hat{\mathbf{x}}(k|k-1). \tag{11}$$

The block diagram shown in Fig. 2 depicts the flowchart of a standard procedure of stochastic signal processing with Kalman filter.

The extended Kalman filter (EKF) is a generalization of the steady-state filter for time-varying systems or linear time-invariant (LTI) systems with a non-stationary noise covariance.

The Kalman filter approach inherently has the flexibility of incorporating the system dynamics equations into the algorithm as well as the provision for uncertainties in the system. Shi et al. [10] used a Kalman filter algorithm to identify the model parameters in the frequency domain. Hoshiya and Saito [12] demonstrated the application of EKF to the problem of identifying system parameters in the frequency domain. Koh et al. [8] presented a condensation method for local damage detection of a multi-storey frame building, in which the remedial stiffness matrix was identified by applying the EKF. Quek et al. [9] identified the variables of state-space equation by EKF, in which the excitation was assumed to be immeasurable, and hence was looked on as a pure white noise input excitation of $\eta(t)$ passing through a filter, thus

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{Bu}(k) = \mathbf{A}\mathbf{x}(k) + \eta(k),\\ \mathbf{y}_v(k) &= \mathbf{C}\mathbf{x}(k) + v(k).\end{aligned} \tag{12}$$

In the next section the Kalman filter will be applied in conjunction with the proposed ARX model for filtering the measured noisy signals for subsequent damage diagnosis purpose.

3. Analysis of the effect of using Kalman filter on the ARX model

In this section the ARX model concerned is examined for its susceptibility to the noise content in the measured signals. The appropriate way of using the Kalman filter to process the noisy signals, given the ultimate objective as being to preserve the state information, and the effectiveness of such an approach are demonstrated mathematically.

In the previous paper [1], an ARX model was established to model the process of a linear system with input as an acceleration at a specific dof and response (measurement) as accelerations at other dofs. The model was ARX (2, 2, 0), with the orders of auto-regressive, exogenous, and delays of exogenous being 2, 2, and 0, respectively. The damage diagnosis is achieved by analyzing the residual errors in the predicted response using the reference model. The ARX model is written as

$$\mathbf{y}(k) = \mathbf{P}_1\mathbf{y}(k-1) + \mathbf{P}_2\mathbf{y}(k-2) + \mathbf{D}\mathbf{u}(k) + \mathbf{E}_1\mathbf{u}(k-1) + \mathbf{E}_2\mathbf{u}(k-2), \tag{13}$$

where both the model output $\mathbf{y}(k)$ and input $\mathbf{u}(k)$ are acceleration responses of the structure. $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{D}$, $\mathbf{E}_1$, $\mathbf{E}_2$ are the model parameters. For undamped systems, these parameters can be explicitly related to the system dynamic properties [1], namely, $\mathbf{P}_1 = 2\mathbf{\Phi}\cos(\mathbf{\Lambda}^{1/2}\Delta t)\mathbf{\Phi}^{\mathrm{T}}$, $\mathbf{P}_2 = -\mathbf{I}$, $\mathbf{D} = \mathbf{L}$, $\mathbf{E}_1 = -\mathbf{\Phi}[\mathbf{I} + \cos(\mathbf{\Lambda}^{1/2}\Delta t)]\mathbf{\Phi}^{\mathrm{T}}\mathbf{L}$, and $\mathbf{E}_2 = \mathbf{\Phi}\cos(\mathbf{\Lambda}^{1/2}\Delta t)\mathbf{\Phi}^{\mathrm{T}}\mathbf{L}$, where $\mathbf{\Lambda}$ is the eigenvalue matrix, $\mathbf{\Phi}$ is the eigenvector matrix, and $\mathbf{L}$ denotes the input coefficient vector. For damped systems, it is difficult to establish an explicit relationship for ARX model coefficients and the system dynamic properties; however, the inherent association of the ARX coefficients with the physical properties still exists [14] and this provides the underlying basis for the construction of a damage feature with the time-domain ARX model.

In order to analyze the noise effect on the system represented by the ARX model, it is advantageous to express the system in a state-space form. This can be done through a standard procedure (see e.g. Ref. [15]). By introducing a variable vector $\mathbf{x}(k)$, the ARX model of Eq. (13) can be expressed in a state-space form, as

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k),\\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k),\end{aligned} \tag{14}$$

where the state parameters $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are related to the ARX model parameters as follows (see Appendix A for the detail working):

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_1 & \mathbf{I} \\ \mathbf{0} & \mathbf{P}_2 & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{D} \\ \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{14a}$$

$$\mathbf{Q}_1 = \mathbf{E}_1 + \mathbf{P}_1\mathbf{D}, \quad \mathbf{Q}_2 = \mathbf{E}_2 + \mathbf{P}_2\mathbf{D}. \tag{14b}$$

The process and measurement noises can be easily incorporated into the above state-space model. Consider firstly a noise term in the measurement $\mathbf{y}$, we have:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k),\\ \mathbf{y}_v(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k),\end{aligned} \tag{15}$$

where $\mathbf{y}_v$ refers to the measured output, $\mathbf{u}$ is supposed to be the true input signal. But as $\mathbf{u}$ in the present model is actually an acceleration response at a particular dof of the structure, only a measured signal with noise, $\mathbf{u}_v$, is available, $\mathbf{u}_v(k) = \mathbf{u}(k) + \boldsymbol{\eta}(k)$, where $\boldsymbol{\eta}(k)$ is the noise content in the measured "input" signal. Consequently, the system should be re-written as

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}_v(k) - \mathbf{B}\boldsymbol{\eta}(k),\\ \mathbf{y}_v(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k).\end{aligned} \tag{16}$$

Herein $\boldsymbol{\eta}(k)$ and $\boldsymbol{\nu}(k)$ are assumed to be white noise processes with zero means and finite covariance of $\sigma(\boldsymbol{\eta}) = \mathbf{Q}$ and $\sigma(\mathbf{v}) = \mathbf{R}$, respectively.

The Kalman filter can now be applied on the model described by Eq. (16). But it is important to note that, because the raw input $\mathbf{u}_v$ has been used in place of the true input $\mathbf{u}$ which is not known, the filtered response signals $\hat{\mathbf{y}}$ is not really an optimal estimate of the true responses $\mathbf{y}$ in Eq. (14). The achieved estimate of the response signals, $\hat{\mathbf{y}} = \hat{\mathbf{y}}(k|k-1)$, is in fact an optimal estimate of the "virtual" response of the system, $\mathbf{y}_{\mathrm{vir}}$, as if the noisy input signal were the "true" input, i.e.,

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}_v(k),\\ \mathbf{y}_{\mathrm{vir}}(k) &= \mathbf{C}\mathbf{x}(k),\end{aligned} \tag{17}$$

where the state parameters $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are those of Eq. (16).

As a matter of fact, here lies the novelty of the present approach. Indeed, in the context of the current damage diagnosis scheme which looks for the deviation from a reference state model, it is not the quality of each individual signal but a system relationship formed by the input–output set that ultimately determines the outcome of the diagnosis. With an inevitable noisy $\mathbf{u}_v$ instead of the true $\mathbf{u}$, what we are looking for is actually a good representation of the process described by Eq. (17) rather than that described by Eq. (16). In what follows, we will demonstrate the $\hat{\mathbf{y}}$ produced from the Kalman filter using $\mathbf{u}_v$ as input is indeed the closest estimate of $\mathbf{y}_{\mathrm{vir}}$ in Eq. (17).

Assuming the "noise" in the virtual measured counterpart for $\mathbf{y}_{\mathrm{vir}}$ of Eq. (17), denoted as $\mathbf{y}_{\mathrm{v,vir}}$, is $\boldsymbol{\gamma}$, it has

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}_v(k), \tag{18a}$$

$$\mathbf{y}_{v,vir}(k) = \mathbf{C}\mathbf{x}(k) + \boldsymbol{\gamma}(k), \tag{18b}$$

$$\mathbf{H}(k) = E(\boldsymbol{\gamma}(k-1)\boldsymbol{\gamma}(k-1)^{\mathrm{T}}). \tag{18c}$$

The application of the Kalman filter on this model delivers the optimal estimator by the following equations:

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{A}\hat{\mathbf{x}}(k-1|k-1) + \mathbf{B}\mathbf{u}_v(k-1), \tag{19a}$$

$$\mathbf{P}(k|k-1) = \mathbf{A}\mathbf{P}(k-1|k-1)\mathbf{A}^{\mathrm{T}}, \tag{19b}$$

$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{M}(k)\left(\mathbf{y}_{v,vir}(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\right), \tag{19c}$$

$$\mathbf{M}(k) = \mathbf{P}(k|k-1)\mathbf{C}^{\mathrm{T}}\left(\mathbf{H}(k) + \mathbf{C}\mathbf{P}(k|k-1)\mathbf{C}^{\mathrm{T}}\right)^{-1}, \tag{19d}$$

$$\mathbf{P}(k|k) = (\mathbf{I} - \mathbf{M}(k)\mathbf{C})\mathbf{P}(k|k-1). \tag{19e}$$

With the same definition of $\mathbf{K}(k)$ in Eq. (9), the recursion of the error covariance matrix is

$$\mathbf{P}(k+1|k) = \mathbf{A}\mathbf{P}(k|k-1)\mathbf{A}^{\mathrm{T}} - \mathbf{K}(k)\mathbf{C}\mathbf{P}(k|k-1)\mathbf{A}^{\mathrm{T}}. \tag{20}$$

The $\mathbf{P}$ in the Kalman filter has the definition as

$$\mathbf{P}(k+1|k) = E[(\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k))(\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k))^{\mathrm{T}}]$$

and

$$\mathbf{P}(k|k) = E[(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))^{\mathrm{T}}].$$

The recursion of the optimal estimator becomes:

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{A}\hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\mathbf{e}(k) + \mathbf{B}\mathbf{u}_v(k), \tag{21a}$$

$$\hat{\mathbf{y}}(k|k-1) = \mathbf{C}\hat{\mathbf{x}}(k|k-1), \tag{21b}$$

$$e(k) = \mathbf{y}_{v,vir}(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1). \tag{21c}$$

The residual error between $\hat{\mathbf{y}}(k+1|k)$ and $\mathbf{y}_{vir}(k+1)$ can be evaluated by combining Eqs. (17) and (21b), as

$$y_{vir}(k+1) - \hat{\mathbf{y}}(k+1|k) = \mathbf{C}(\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k)). \tag{22a}$$

By Eqs. (18a) and (21a) the error for the state vector is

$$\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k) = \mathbf{A}(\mathbf{x}(k+1) - \hat{\mathbf{x}}(k|k-1)) - \mathbf{K}(k)\mathbf{e}(k). \tag{22b}$$

Following the definition of $\mathbf{P}$ in the Kalman filter, the covariance error matrix of Eqs. (22a) and (22b) is

$$E\left[\left(\mathbf{y}_{vir}(k+1) - \hat{\mathbf{y}}(k+1|k)\right)\left(\mathbf{y}_{vir}(k+1) - \hat{\mathbf{y}}(k+1|k)\right)^{\mathrm{T}}\right] = \mathbf{C}\mathbf{P}(k+1|k)\mathbf{C}^{\mathrm{T}}, \tag{23a}$$

$$E\left[(\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k))(\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k))^{\mathrm{T}}\right] = \mathbf{P}(k+1|k). \tag{23b}$$

In these expressions, $\mathbf{x}(k)$ and $\mathbf{y}(k)$ denote the real values for the state vector $\mathbf{x}$ and output $\mathbf{y}$ at kth time step, while $\hat{\mathbf{x}}(k|k-1)$ and $\hat{\mathbf{y}}(k|k-1)$ denote the prediction values for $\mathbf{x}$ and $\mathbf{y}$, respectively. It has been demonstrated by Harvey [13] that $\hat{\mathbf{x}}(k|k-1)$ is the minimum mean square linear estimator of $\mathbf{x}(t)$ based on observations $\mathbf{u}_v(k)$ and $\mathbf{y}_v(k)$ up to time $k-1$. This estimator is unconditionally unbiased and the unconditional covariance matrix of the estimation error is the $\mathbf{P}(k|k-1)$ given by the Kalman filter. Therefore, it can be concluded that the $\hat{\mathbf{y}}(k|k-1)$ herein is the closest estimator to $\mathbf{y}_{\mathrm{vir}}(k)$, as the error of ($\hat{\mathbf{y}}(k|k-1)$-$\mathbf{y}_{\mathrm{vir}}(k)$) is directly related to the error of ($\hat{\mathbf{x}}-\mathbf{x}$).

Unfortunately it is usually difficult to obtain an explicit solution of $\mathbf{P}(k+1|k)$; hence it is difficult to evaluate how close the $\hat{y}(k|k-1)$ is to $\boldsymbol{y}_{\mathrm{vir}}(k)$ by an explicit equation.

4. Implementation of the Kalman filter in the ARX based damage diagnosis

4.1. ARX model estimation and signal processing using Kalman filter

When the signals $\mathbf{y}(k)$ and $\mathbf{u}(k)$ are contaminated with noise, the least-squares algorithm of an ARX model may lead to an inaccurate estimation of the model parameters. A common method in overcoming the noise influence on the model estimation is using the approximate maximum likelihood (ML) estimators to estimate the model parameters, based on the Kalman filter [13]. For a stable system, the error covariance $\mathbf{P}$ will converge such that the Kalman filter eventually has a time invariant solution. This solution can be transformed into an ARMAX model [15], which can be directly used for the model estimation and prediction to the same effect as directly using the Kalman filter. Further elaboration follows.

For a stable system, the error covariance $\mathbf{P}$ and the Kalman gain $\mathbf{K}$ converage to the steady-state values $\bar{\mathbf{P}}$ and $\bar{\mathbf{K}}$ as $k \to \infty$ [13], i.e., $\lim_{t\to\infty}\mathbf{P}(t+1|t) = \bar{\mathbf{P}}$ and $\lim_{t\to\infty}\mathbf{K}(k) = \bar{\mathbf{K}}$. Specializing the Kalman filter on the model of Eq. (21), it can be written as

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{A}\hat{\mathbf{x}}(k-1|k-2) + \bar{\mathbf{K}}\,\mathbf{e}(k-1) + \mathbf{B}\mathbf{u}_v(k-1), \tag{24a}$$

$$\hat{\mathbf{y}}(k|k-1) = \mathbf{C}\hat{\mathbf{x}}(k|k-1), \tag{24b}$$

$$\mathbf{e}(k-1) = \mathbf{y}_v(k-1) - \mathbf{C}\hat{\mathbf{x}}(k-1|k-2). \tag{24c}$$

By taking the Z-transform of Eqs. (24a, 24b), the following equation can be obtained [14]:

$$\begin{aligned}\hat{\mathbf{y}}(k+1|k) = {} & \mathbf{P}_1\hat{\mathbf{y}}(k|k-1) + \mathbf{P}_2\hat{\mathbf{y}}(k-1|k-2) + \mathbf{D}\mathbf{u}_v(k) + \mathbf{E}_1\mathbf{u}_v(k-1) + \mathbf{E}_2\mathbf{u}_v(k-2) \\ & + \mathbf{K}_1\mathbf{e}(k) + (\mathbf{K}_2 - \mathbf{P}_1\mathbf{K}_1)\mathbf{e}(k-1) + (\mathbf{K}_3 - \mathbf{P}_2\mathbf{K}_1)\mathbf{e}(k-2).\end{aligned} \tag{25}$$

It can be further written as an ARMAX model:

$$\begin{aligned}\mathbf{y}_v(k+1) = {} & \mathbf{P}_1\mathbf{y}_v(k) + \mathbf{P}_2\mathbf{y}_v(k-1) + \mathbf{D}\mathbf{u}_v(k) + \mathbf{E}_1\mathbf{u}_v(k-1) + \mathbf{E}_2\mathbf{u}_v(k-2) \\ & + \mathbf{e}(k+1) + \mathbf{H}_1\mathbf{e}(k) + \mathbf{H}_2\mathbf{e}(k-1) + \mathbf{H}_3\mathbf{e}(k-2),\end{aligned} \tag{26}$$

where

$$\mathbf{E}_1 = \mathbf{Q}_1 - \mathbf{P}_1\mathbf{D}, \quad \mathbf{E}_2 = \mathbf{Q}_2 - \mathbf{P}_2\mathbf{D}, \tag{26a}$$

$$\mathbf{H}_1 = \mathbf{K}_1 - \mathbf{P}_1, \quad \mathbf{H}_2 = \mathbf{K}_2 - \mathbf{P}_1\mathbf{K}_1 - \mathbf{P}_2, \quad \mathbf{H}_3 = \mathbf{K}_3 - \mathbf{P}_2\mathbf{K}_1, \tag{26b}$$

$$\bar{\mathbf{K}} = \begin{bmatrix}\mathbf{K}_1 \\ \mathbf{K}_2 \\ \mathbf{K}_3\end{bmatrix}. \tag{26c}$$

The process parameter matrices $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{D}$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ are identical to those in Eqs. (13) and (14). The prediction $\hat{\mathbf{y}}(k+1|k)$ here is the same as expressed by the steady-state Kalman filter.

For stable systems under consideration, the ARMAX model of Eq. (26) can be used to fulfill the function of the Kalman filter in a more straightforward way. It has been found that a method utilizing ARMAX model usually yields good estimations for the model parameters even under the condition of a relatively large noise level [3,5,15]. A least-square approach can be applied to determine the model coefficients [$\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{D}$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$] of the ARMAX model (and hence the state parameters $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ according to Eqs. (14a) and (14b)) . The block diagram of Fig. 3 shows the flowchart of the standard procedure of the model estimation with the ARMAX model. Some trial numerical simulations have been performed, and the results confirmed the satisfactory accuracy of using the ARMAX model in the estimation of the state parameters.

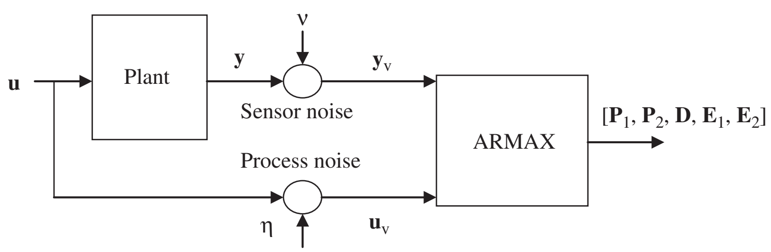


Fig. 3. Standard procedure of model estimation with ARMAX model.

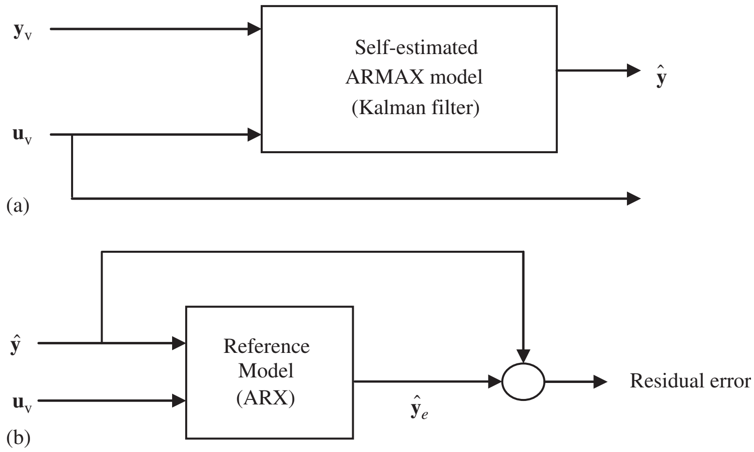


Fig. 4. Procedure of damage diagnosis.

4.2. Implementation for damage diagnosis

The block diagram in Fig. 4 shows the implementation procedure of the Kalman filter (ARMAX model) on the measured input–output signals and the subsequent damage diagnosis operations. It can be summarized into the following essential steps:

(i) Select the "input" acceleration signal (a response at a particular dof) and the output "response" acceleration signals as outlined in Ref. [1].
(ii) For a selected reference state of the structure, estimate the model parameters using the ARMAX scheme on the corresponding set of signals. Dropping the moving average (MA) part gives rise to the reference ARX model.
(iii) For any other state of the structure, acquire the input-response signals. Establish the current ARMAX model from the set of signals acquired. Then perform the stochastic processing of the signals through the current ARMAX model (Kalman filter) with the raw $\mathbf{u}_v(k)$ as input, get the virtual input-response set.
(iv) Apply the reference ARX model established in step (i) on the above virtual input-response set, and analyze the residual error of the ARX results and assess the damage using an appropriate residual error feature
(v) Repeat step (ii) to (iv) for any other state of the structure requiring diagnosis

For the diagnosis from the residual error, herein a statistical feature called CRE is employed to indicate damage. The CRE is defined as the percentage of the error variation, as

$$\mathrm{CRE} = \sigma(e)/\sigma(y) \times 100\%, \tag{27}$$

where $\sigma(e)$ is the covariance of the residual error when the reference model is applied on the current-state signals (Step iv in the above procedure), and $\sigma(y)$ is the covariance of the output $\mathbf{y}(k)$. This feature indicates the percentage of unfitness of the signal to the model, and thus serves as a relative indicator of the changes in the underlying dynamic system when compared with the CRE of the reference state.

In the next section some numerically simulated examples and an experimental study will be given to illustrate the implementation procedure and the effectiveness of the approach in reducing the effect of noise on the residual error for the damage diagnosis.

5. Numerical examples

In this section the effectiveness of the model under noisy condition is studied using numerical simulation. A two-dof mass spring model, shown in Fig. 5, is considered here for the numerical simulation study. Each point mass is 419.4 kg and the initial spring stiffness $k1$ and $k2$ are both 56.7 kN m^{-1}. The damping ratio is assumed to be 0.05. The natural periods of the system are calculated to be $T_1 = 0.874$ s and $T_2 = 0.334$ s.

The excitation on the structure is imposed at the base by random acceleration. The unit of the input time series is m/s^2. Two types of excitation signals are considered, one is Gaussian White noise; another is a general random noise generated from normally distributed numbers with a given standard deviation. Three random series are used, namely, (a) Wgn1 (White noise) with a standard deviation of 1.13, (b) Randn 1 (random noise) with a standard deviation of 1.0. The sampling time step is 0.01 s and the number of data points is chosen to be 4000 for each sample piece. The same time step and duration are used in recording the response signals from the simulation. The simulated acceleration responses are designated according to their excitation series, e.g., Wgn 1 and Randn 1.

The noise components $\eta(k)$ and $v(k)$ are added into the responses after the generation of the acceleration responses at $m1$ and $m2$ by the numerical simulations. In accordance with the assumption stated earlier, the artificial noises $\eta(k)$ and $v(k)$ are white noise, with zero mean and the covariance $E = (\eta\eta^{\mathrm{T}}) = Q$, $E(vv^{\mathrm{T}}) = R$, $E(\eta v^{T}) = 0$.

5.1. Model estimation

The ARX(2,2,0) model of the two-dof mass spring system can be expressed as

$$\mathrm{A}(q)y(k) = B(q)u(k).$$

Here $y(k)$ is defined as the response at $m2$ and $u(k)$ is the response at $m1$. q^{-1} is the delay operator.

A least-squares approach is applied to determine the model parameters. The procedure is performed using Matlab system identification toolbox. When the noise-free signal are used, the model parameters are estimated as

$$A(q) = 1 - 1.939\,q^{-1} + 0.9794\,q^{-2}, \tag{28a}$$

$$B(q) = 0.9952 - 1.948\,q^{-1} + 0.9798\,q^{-2}. \tag{28b}$$

This model can be looked on as the exact model of the system. It gives a covariance of the residual error with respect to the pure signal as small as 1.0×10^{-5}.

When 10% noise ($E(vv^{\mathrm{T}}) = 0.01E(yy^{\mathrm{T}})$) is added into the output signal $y(k)$, the ARX(2,2,0) model estimated from the noisy reference signal becomes:

$$A(q)y(k) = B(q)u(k),$$

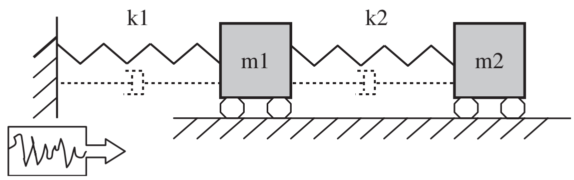


Fig. 5. A two-dof mass-spring system.

$$A(q) = 1 - 1.387q^{-1} + 0.4391q^{-2},$$

$$B(q) = 0.9591 - 1.389\,q^{-1} + 0.4549\,q^{-2}.$$

With this model the covariance of the residual error is around 0.5. Comparing with the "exact" model shown in Eqs. 28(a) and 28(b), the coefficients estimated by the noisy reference signals deviate significantly from the correct values. Consequently, the model will not predict the signal to a satisfactory accuracy. This example demonstrates that a direct model estimation by the least-squares approach is unable to produce an accurate ARX(2,2,0) model from noisy reference signals.

The ARMAX model, as mentioned in Section 4.1, is now considered. The model is ARMAX(2,3,3,0) (auto-regressive, exogenous and moving average of orders 2, 3, and 3, respectively, and delays of exogenous 0). The model is estimated from the signal with 50% noise ($E(vv^{\mathrm{T}}) = 0.25E(yy^{\mathrm{T}})$) in the output signal $y(k)$, as

$$A(q)y(t) = B(q)u(t) + C(q)e(t),$$

$$A(q) = 1 - 1.94q - 1 + 0.9803q^{-2},$$

$$B(q) = 0.9945 - 1.948q^{-1} + 0.9802q^{-2},$$

$$C(q) = 1 - 1.903q^{-1} + 0.9061q^{-2} + 0.0379q^{-3}.$$

It can be observed that the coefficients $A(q)$ and $B(q)$ are very close to the "exact" ARX model in Eqs. 28(a) and 28(b). This shows that the coefficients of the ARX model can be estimated quite accurately even when the noise content in the signal is high.

To further illustrate the influence of the noise content on the model estimation and the performance of the estimated model, several noise scenarios of reference signals are explored, namely (refer to Table 1, first column): (1) white noise excitation, noise-free signals; (2) white noise excitation, 10% noise in the model output (acceleration at m2) only; (3) random excitation, 10% noise in the model output only; (4) white noise excitation, 10% noise in both the model input (acceleration at m1) and output signals; and (5) white noise excitation, 20% noise in both the model input and output signals. From each set of the reference signals, the ARX model parameters are estimated using the ARMAX scheme as mentioned above. Subsequently, three sets of test signals with no noise, 10% noise and 20% noise, respectively, as indicated in Table 1, are applied on each of the above five reference ARX models. The errors in terms of CRE are summarized in Table 1.

From Table 1 it can be seen that the errors are closely correlated to the level of noise in the test signals (viewing the table row-wise), but are relatively independent of the scenarios from which the reference model has been established (viewing the table column-wise). The interpretations can be two folds, (1) the reference models are estimated satisfactorily using the ARMAX scheme; and (2) the performance of the reference model thus depends primarily on the quality of the current signals supplied for diagnosis. When the current signals contain a high noise level, the residual errors can be high even the structural state remain unchanged.

Table 1
CRE(%) by reference models for different noise level

Reference models	Testing signals under various noise levels		
	Randn1, noise-free signal	Wgn1, 10% noise output +10% input	Randn1, 20% noise output+20% input
Wgn1, Noise free signal	8.63e-4	34.24	70.56
Wgn1, 10% noise in output	0.41	34.18	70.50
Randn1, 50% noise in output	1.22	34.09	70.40
Wgn1, 10% noise output +10% input	3.21	33.93	70.12
Randn1, 20% noise output+20% input	2.36	34.26	69.7

5.2. Noise filtering (stochastic processing) of signals

Following the procedure described in Section 4.2, the three sets of test signals are processed through the Kalman filter (the ARMAX model derived from each set itself) to overcome the noise effect. The processed sets of input (raw u_v) and response ($\hat{\mathbf{y}}$) signals are then applied on the reference models, and the predicted $\hat{\mathbf{y}}_e$ are compared with the above $\hat{\mathbf{y}}$ as the residual error (refer to Fig. 4). For an observation, three reference models, designated as W1, W2 and R1, are established respectively from the signals corresponding to Wgn1 with two different noise levels and Randn1 with a 20% noise level. Table 2 summarizes the resulting residual errors in terms of CRE for the three reference models. Comparing with the corresponding results in Table 1, it can be seen that the error feature (CRE) are drastically reduced, from as much as 70% without the "filtering" process to generally below 3% using the "filtered" signals. This level of accuracy for the same state of the structure indicates that the approach should be workable to diagnose sensible damages.

Now two damage scenarios are introduced in for testing the diagnosis ability of the approach, including (a) a stiffness reduction of 20% on $k2$, (b) a stiffness reduction of 10% on $k2$. The same three reference models as mentioned in the previous paragraph are examined. Three different sets of test signals from each of the two damage scenarios are considered. The test signals are processed first through the Kalman filter and then applied on the reference models. The results of the residual errors in terms of CRE are summarized in Tables 3 and 4, respectively, and the representative error results are also plotted in Fig. 6.

It can be observed that the residual error feature (CRE) for the same damage scenario are very consistent among those with different levels of noise. The "filtering" process renders the results in terms of CRE from noisy measurements to be almost as good as that from totally noise-free signals. Furthermore, the CRE value shows a good correlation with the degree of damage; the CRE is about 23% for 20% stiffness reduction, and about 14% for 10% stiffness reduction.

The above numerically simulated scenarios show clearly the effectiveness of the proposed approach depicted in Fig. 4 in using noisy time-domain acceleration signals for the diagnosis of damage. In the next section the approach is further tested on a physical experiment case.

Table 2
CRE(%) from reference signals with different noise levels after implementation of the proposed procedure

Reference models	Testing signals under various noise levels		
	Noise-free signal	Wgn1, 10% noise output + 10% input	Randn1, 20% noise output + 20% input
Wgn1 Noise free signal (Model **W1**)	8.63e-4	3.17	2.65
Wgn1 10% noise output + 10% input (Model **W2**)	3.21	1.40	2.98
Randn 1 20% noise output + 20% input (Model **R1**)	2.36	2.58	0.53

Table 3
CRE(%) from processed signals of damage scenario (a)

Reference models	Testing signals under various noise levels		
	Noise-free signal	Wgn1 10% noise output + 10% input	Wgn1 20% noise output + 20% input
Noise free signal	23.69	23.46	24.72
Wgn1 10% noise output + 10% input	23.31	23.03	24.08
Randn1 20% noise output + 20% input	22.92	22.56	23.18

Table 4
CRE (%) from processed signals of damage scenario (b)

Reference models	Testing signals under various noise levels		
	Noise-free signal	Wgn1, 10% noise output + 10% input	Wgn1, 20% noise output + 20% input
Noise free signal	13.75	14.27	15.35
Wgn1 10% noise output + 10% input	13.32	13.71	14.54
Randn1 20% noise output + 20% input	13.13	13.32	13.53

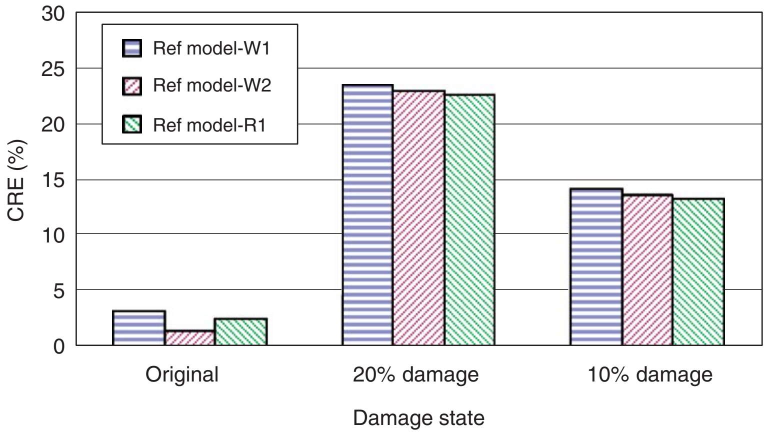


Fig. 6. Damage feature (CRE) for three states of structure based on stochastically processed virtual input-response set.

6. Experimental example

A two-storey concrete model frame was fabricated and tested on a shake table to different degrees of damage at multiple stages. Fig. 7 shows the test setup. Low-amplitude random vibration tests were conducted before and after each major test to get the vibration signals, and these measurements were then used for damage diagnosis purpose. In what follows, the acceleration signals acquired from the complementary random vibration tests for the initial state and an intermediate state with a degree of damage equivalent to about 10% stiffness reduction [16] are analyzed using the proposed approach. The sampling rate for the data acquisition was 1000 Hz.

Table 5 lists the four sets of random vibration response signals taken from the test data for the present application. S1-1 and S1-2 are two sets of signals from the original state (state "S1") of the structure, while S2-1 and S2-2 are two sets of signals from a damaged state (state "S2") of the structure. For a cross comparison, two reference ARX models are established, using signals from the original state (S1) and the damaged state of the structure (S2), respectively.

The residual error CREs after implementing the procedure of Fig. 2 (without "filtering") and of Fig. 4 (with "filtering"), respectively, are plotted in Fig. 8. For the cases without "filtering" the signals (Fig. 8a), the residual error CREs do not show clear changes of the state of the structure from the reference state. However, from the error CRE results shown in Fig. 8b) after implementing the filtering operation, it can be clearly observed a change of the state of the structure as compared to the reference state. For example, when the reference state is established from S1 signals, around 8% CRE is obtained for another set of signals from the same state of the structure, whereas as much as 20% CRE is observed for a set of signals from the damage state of the structure. Vise versa, when the reference

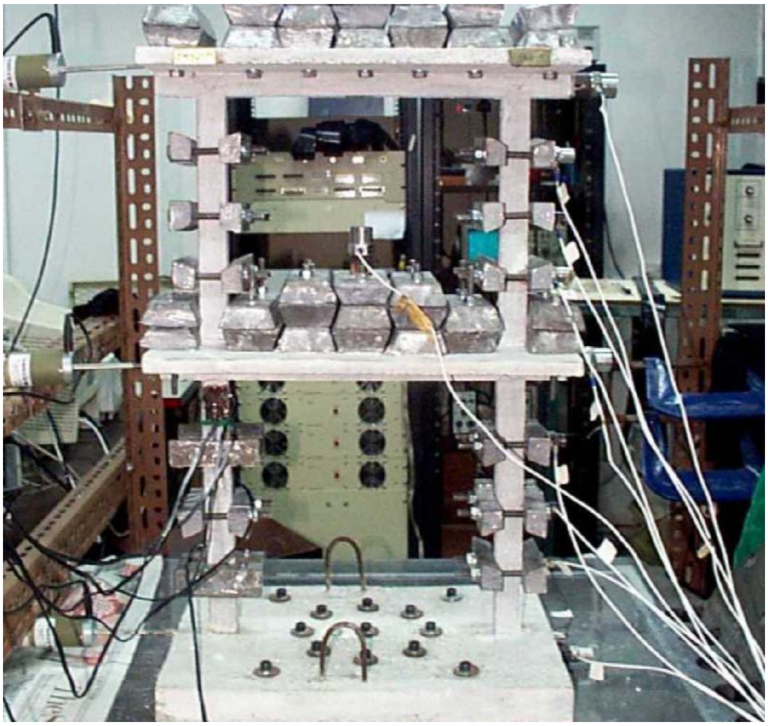


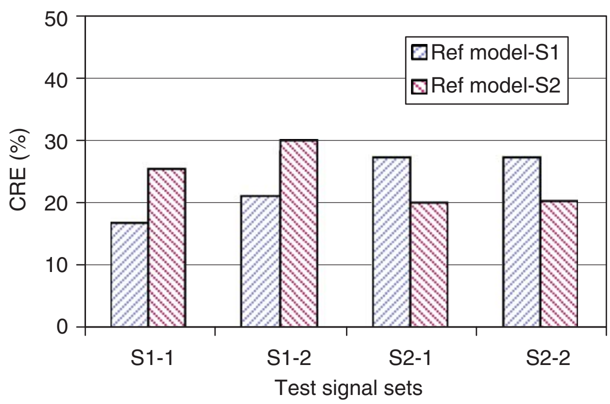
Fig. 7. Shake-table test of two-storey concrete frame model.

Table 5
Summary of measured signals from the test frame

Name of time series	Description	Power of base excitation (STD)	Power of floor-1 acceleration (STD)	Power of floor-2 acceleration (STD)
S1-1	Initial state, time series 1	0.0636	0.0384	0.0444
S1-2	Initial state, time series 2	0.0651	0.0349	0.0366
S2-1	Damage state, time series 1	0.0475	0.0311	0.0342
S2-2	Damage state time series 2	0.0497	0.0317	0.0408

state is established using the damage state signals, the CRE is about 10% for another set of signals form the same state of the structure and increases to about 25% for signals from the original state of the structure.

It should be mentioned that the results for this particular experimental case are not as good as in the previous numerical simulated scenarios. The reason is believed not primarily because of the quality of the measured signals; but rather because of some complication of the test model configuration which involved additional masses that served for the purpose of other test requirements [16]. Further experimental verification of the approach can be carried out with other relevant experimental data.



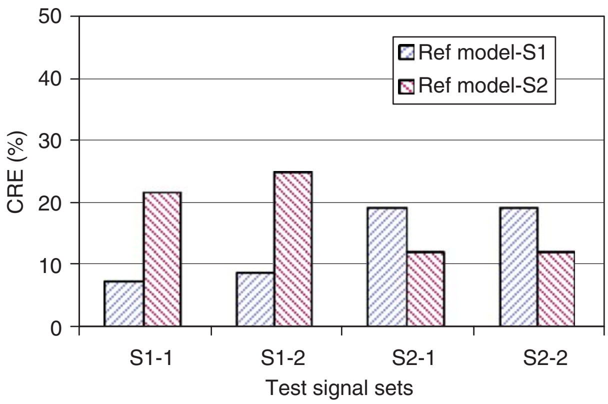


Fig. 8. Residual error feature (CRE) before (left) and after (right) implementing the proposed procedure.

7. Conclusions

A procedure based on Kalman filter is presented for the time-domain analysis of noise-contaminated vibration signals for the structural damage diagnosis. The approach stems from a previously proposed ARX model, in which the acceleration signals measured at various degrees of freedoms of a structure forms the basic input-response set. The Kalman filter is incorporated to perform the stochastic processing of the input-response signals containing noise. It is demonstrated that using the raw input signal and the processed response signals to form the virtual input-response set, the underlying physical state of the structure is well preserved. By presenting the above virtual input-response signals to the reference model, the residual error becomes reasonably indicative to the degree of damage in the structure. The effectiveness of the proposed approach is demonstrated using numerically simulated scenarios and an experimental example. It is shown that using the proposed procedure the effect of noise on the outcome of the damage diagnosis is minimized, and the residual error feature in terms of CRE using the processed set of virtual input-response signals correlate well with the severity of damage.

It should be noted that at present the CRE feature does not have an absolute threshold for damage. It needs to be interpreted with respect to the CRE of the reference state error, and it is presumably also dependent on the inherent sensitivity of the measured response signals to the structural damage. For simple systems as in the example of the paper, the percentage increase of the CRE generally correlate to the same order of percentage stiffness change in a major component of the system. For complex systems, the threshold or the general correlation of CRE to structural changes may need to be explored on individual case basis, along with the selection of the measurement dofs.

Appendix A

Transformation of ARX(2, 2, 0) to state space model Eq. (14)

The ARX(2, 2, 0) model is written as

$$\mathbf{y}(k) = \mathbf{P}_1\mathbf{y}(k-1) + \mathbf{P}_2\mathbf{y}(k-2) + \mathbf{D}\mathbf{u}(k) + \mathbf{E}_1\mathbf{u}(k-1) + \mathbf{E}_2\mathbf{u}(k-2) \tag{13}$$

Introduce a variant vector $\mathbf{x}(k)$

$$\mathbf{x}(k) = \begin{Bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ \mathbf{x}_3(k) \end{Bmatrix}$$

such that $\mathbf{x}(k)$ satisfies the following four equations [14]:

$$\mathbf{y}(k-1) = \mathbf{x}_1(k), \tag{a}$$

$$\mathbf{x}_1(k) = \mathbf{x}_2(k-1) + \mathbf{Du}(k-1), \tag{b}$$

$$\mathbf{x}_2(k) = \mathbf{P}_1\mathbf{x}_2(k-1) + \mathbf{x}_3(k-1) + (\mathbf{E}_1 + \mathbf{P}_1\mathbf{D})\mathbf{u}(k), \tag{c}$$

$$\mathbf{x}_3(k) = \mathbf{P}_2\mathbf{x}_2(k-1) + (\mathbf{E}_2 + \mathbf{P}_2\mathbf{D})\mathbf{u}(k-1). \tag{d}$$

Here we notice the $x_1(k)$ is one step ahead of $y(k)$, but in order to give a standard form of the state space model, we write

$$\mathbf{y}(k) = \mathbf{Cx}(k) \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

This implies that when the model is applied, the $\mathbf{y}(k)$ should be shifted one step ahead, and add $\mathbf{y}(1) = \{0\}$.

Then we get the state-space model

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{Ax}(k) + \mathbf{Bu}(k), \\ \mathbf{y}(k) &= \mathbf{Cx}(k), \end{aligned} \tag{14}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_1 & \mathbf{I} \\ \mathbf{0} & \mathbf{P}_2 & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{D} \\ \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{14a}$$

$$\mathbf{Q}_1 = \mathbf{E}_1 + \mathbf{P}_1\mathbf{D}, \quad \mathbf{Q}_2 = \mathbf{E}_2 + \mathbf{P}_2\mathbf{D}. \tag{14b}$$

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